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Structural Complexity of Context-Free Languages

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A language L_1 is said to be fbt-translatable to a language L_2 if there exists a functional binary transduction f such that $L_1 \subseteq$ domain (f), $f(L_1) \subseteq L_2$ and $f(\bar{L}_1) \subseteq \bar{L}_2$. L_1, L_2 are said to be fbt-equivalent if each of them is fbt-translatable to the other. The fbt-translatability relation is naturally extended to a partial ordering in the set of all fbt-equivalence classes.

The main results are: (1) There is no least fbt-equivalence class in the set of all fbt-equivalence classes of nonregular context-free languages; (2) The fbt-translatability relation in the set of all fbt-equivalence classes of context-free languages is not a lattice.

INTRODUCTION

In order to study structural complexity of formal languages, we introduced the notions of the fbt-translatability relation and the fbt-equivalence relation between formal languages (Kobayashi, 1969). A language L_1 is said to be fbt-translatable to another language L_2 if there exists a functional binary transduction (Elgot and Mezei, 1965) f such that (1) $L_1 \subseteq$ domain (f), (2) $f(L_1) \subseteq L_2$, (3) $f(\bar{L}_1) \subseteq \bar{L}_2$. Two languages L_1, L_2 are said to be fbt-equivalent if each of them is fbt-translatable to the other. The class of functional binary transductions seems to be the largest among the classes of transformations that are defined by (possibly nondeterministic) finite state machines (in a possibly implicit way). Hence we may regard an fbt-equivalence class as abstracting one structural feature of formal languages that remains unchanged under these transformations. The fbt-translatability relation is naturally defined between fbt-equivalence classes and gives one means to compare complexities of these structural features of formal languages that have been thus abstracted as fbt-equivalence classes.

In the previous paper (Kobayashi, 1969), we considered the set of all fbt-equivalence classes of context-free languages and showed that the set contains a subset that is considerably complex with respect to the fbt-translatability relation. In the present paper we continue to investigate this set

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and show that (1) there is no least fbt-equivalence class in the set of all fbt-equivalence classes of nonregular context-free languages (Corollary 3), (2) the fbt-translatability relation in the set of all fbt-equivalence classes of context-free languages is not a lattice (Corollary 4).

1. Preliminaries

Let Σ be a fixed denumerably infinite set. An element of Σ is called a *letter*. Let Σ^* denote the set of all finite sequences of elements in Σ including the empty sequence ϵ. An element of Σ^* is called a *word*. A nonempty finite subset of Σ is called an *alphabet*. For an alphabet Σ_1, let $\Sigma_1{}^*$ denote the set of all words containing no letter in $\Sigma - \Sigma_1$. A subset L of Σ^* is called a *language* if there exists an alphabet Σ_1 such that $L \subseteq \Sigma_1{}^*$. A subset R of $\Sigma^* \times \Sigma^*$ is called a *word relation* if there exist alphabets Σ_1, Σ_2 such that $R \subseteq \Sigma_1{}^* \times \Sigma_2{}^*$. For a word relation R, let R^{-1} denote the word relation $\{(v, u) \mid (u, v) \text{ in } R\}$. Let domain (R), range (R) denote the sets $\{u \mid \text{there exists } v \text{ such that } (u, v) \text{ is in } R\}$, $\{v \mid \text{there exists } u \text{ such that } (u, v) \text{ is in } R\}$, respectively. A word relation f is called a *partial word function* if, for every u, there exists at most one v such that (u, v) is in f. For a partial word function f and u in domain (f), let $f(u)$ denote the uniquely determined v such that (u, v) is in f.

A *generalized 2-input nondeterministic finite automaton* (abbreviated *generalized 2NDFA*) is a 6-tuple $A = (S, \Sigma_1, \Sigma_2, \delta, s_0, F)$, where

(1) S is a nonempty finite set (of *states*),

(2) Σ_1, Σ_2 are alphabets (of *inputs* and *outputs*, respectively),

(3) δ is a finite set of elements of the form (s, M, N, s'), where s, s' are in S and M, N are regular subsets of $\Sigma_1{}^*$, $\Sigma_2{}^*$, respectively (the set of *transitions*),

(4) s_0 is an element of S (the *initial state*),

(5) F is a subset of S (the set of *accepting states*).

A nonempty finite sequence of elements of δ of the form

$$(s_{j_0}, M_1, N_1, s_{j_1})(s_{j_1}, M_2, N_2, s_{j_2}) \cdots (s_{j_{n-1}}, M_n, N_n, s_{j_n})$$

is called a *path* of A from s_{j_0} to s_{j_n} ($n \geqslant 1$). Each word u in $M_1 M_2 \cdots M_n$ is called an *input word* of the path and each word v in $N_1 N_2 \cdots N_n$ is called an *output word* of the path. The word relation $\{(u, v) \mid$ there is a path from s_0

to a state in F such that u is an input word and v is an output word of the path} is called the *word relation defined by* A and is denoted by $R(A)$. A generalized 2NDFA $A = (S, \Sigma_1, \Sigma_2, \delta, s_0, F)$ is called a *2-input nondeterministic finite automaton* (abbreviated *2NDFA*) if each element of δ is of the form $(s, \{u\}, \{v\}, s')$ with u in $\Sigma_1{}^*$, v in $\Sigma_2{}^*$. It is easily shown that a word relation is defined by a generalized 2NDFA if and only if it is defined by a 2NDFA (Elgot and Mezei, 1965). A word relation is called a *binary transduction* if it is defined by a 2NDFA. A binary transduction that is a partial word function is called a *functional binary transduction.*

Let L_1, L_2 be languages. L_1 is said to be fbt-*translatable* to L_2 *via* a functional binary transduction f (abbreviated $L_1 \leqslant_{\text{fbt}} L_2$ (via f)) if (1) $L_1 \subseteq$ domain (f), (2) for each u in L_1, $f(u)$ is in L_2, (3) for each u in $\Sigma^* - L_1$, if it is in domain (f), then $f(u)$ is in $\Sigma^* - L_2$. L_1 is said to be fbt-*translatable* to L_2 (abbreviated $L_1 \leqslant_{\text{fbt}} L_2$) if there exists a functional binary transduction f such that $L_1 \leqslant_{\text{fbt}} L_2$ (via f). L_1 is said to be fbt-*equivalent* to L_2 (abbreviated $L_1 \equiv_{\text{fbt}} L_2$) if $L_1 \leqslant_{\text{fbt}} L_2$ and $L_2 \leqslant_{\text{fbt}} L_1$. We write $L_1 <_{\text{fbt}} L_2$ if $L_1 \leqslant_{\text{fbt}} L_2$ but not $L_2 \leqslant_{\text{fbt}} L_1$. The fbt-equivalence is in fact an equivalence relation. An fbt-equivalence class (i.e., the set of all languages that are fbt-equivalent to one specific language) is called an fbt-*degree*. Let γ_1, γ_2 be fbt-degrees and L_1, L_2 be languages in γ_1, γ_2, respectively. γ_1 is said to be fbt-*translatable* to γ_2 and we write $\gamma_1 \leqslant_{\text{fbt}} \gamma_2$ if $L_1 \leqslant_{\text{fbt}} L_2$. We write $\gamma_1 <_{\text{fbt}} \gamma_2$ if $L_1 <_{\text{fbt}} L_2$. These definitions do not depend on the specific choice of L_1, L_2. The fbt-translatability relation is a partial ordering in the set of all fbt-degrees. Let $\mathscr{L}$ be a class of languages. An fbt-degree γ is said to be an fbt-degree of $\mathscr{L}$ if γ contains at least one language of $\mathscr{L}$. For the motivation of the definition of the fbt-translatability relation, see Kobayashi (1969).

Now we give three theorems that are used in the following sections.

THEOREM 1. *$L_1 \leqslant_{\text{fbt}} L_2$ if and only if there exists a regular set M such that $L_1 \equiv_{\text{fbt}} L_2 \cap M$.*

Proof. Suppose $L_1 \leqslant_{\text{fbt}} L_2$ (via f), where f is a functional binary transduction. Let $M =$ range (f). Evidently, M is a regular set and $L_1 \leqslant_{\text{fbt}} L_2 \cap M$ (via f). Although the word relation f^{-1} is a binary transduction, it may not be a functional binary transduction. But Theorem 3 in Kobayashi (1969) shows that there exists a functional binary transduction f' such that $f' \subseteq f^{-1}$ and domain $(f') =$ domain (f^{-1}). Then $L_2 \cap M \leqslant_{\text{fbt}} L_1$ (via f'). Hence $L_1 \equiv_{\text{fbt}} L_2 \cap M$.

Conversely, suppose M is a regular set such that $L_1 \equiv_{\text{fbt}} L_2 \cap M$. Let f be the functional binary transduction $\{(u, u) \mid u \text{ is in } M\}$. Then $L_2 \cap M \leqslant_{\text{fbt}} L_2$

(via f). Hence $L_1 \leqslant_{\mathrm{fbt}} L_2 \cap M \leqslant_{\mathrm{fbt}} L_2$. By the transitivity of the fbt-translatability, we have $L_1 \leqslant_{\mathrm{fbt}} L_2$. Thus, the theorem is proved.

Let $L_1, L_2, \ldots, L_m$ be languages. $L_1, L_2, \ldots,$ are said to be *regularly separable* if there exist mutually disjoint regular sets $M_1, M_2, \ldots, M_m$ such that $L_i \subseteq M_i$ for each i $(1 \leqslant i \leqslant m)$.

The following theorem is useful in determining or estimating the fbt-degree of a given language.

THEOREM 2. *Let $L_1, L_2, \ldots, L_m, L_1', L_2', \ldots, L_n'$ be languages such that* (1) *$L_1, \ldots, L_m$ are regularly separable,* (2) *$L_1', \ldots, L_n'$ are regularly separable,* (3) *for each $i(1 \leqslant i \leqslant m)$ there exists $j(1 \leqslant j \leqslant n)$ such that $L_i \leqslant_{\mathrm{fbt}} L_j'$. Then $L_1 \cup \cdots \cup L_m \leqslant_{\mathrm{fbt}} L_1' \cup \cdots \cup L_n'$.*

Proof. Let $M_1, \ldots, M_m$ be mutually disjoint regular sets such that $L_i \subseteq M_i$ for each $i(1 \leqslant i \leqslant m)$ and let $M_1', \ldots, M_n'$ be mutually disjoint regular sets such that $L_j' \subseteq M_j'$ for each $j(1 \leqslant j \leqslant n)$. For each $i(1 \leqslant i \leqslant m)$, let j_i be an integer and let f_i be a functional binary transduction such that $L_i \leqslant_{\mathrm{fbt}} L_{j_i}'$ (via f_i). Let f be the functional binary transduction $\bigcup_{1 \leqslant i \leqslant m} (f_i \cap M_i \times M_{j_i}')$. Then, $L_1 \cup \cdots \cup L_m \leqslant_{\mathrm{fbt}} L_1' \cup \cdots \cup L_n'$ (via f).

COROLLARY 1. *Let $L_1, \ldots, L_i, \ldots, L_j, \ldots, L_m$ be regularly separable and $L_i \leqslant_{\mathrm{fbt}} L_j$. Then*

$$\begin{aligned} &L_1 \cup \cdots \cup L_i \cup \cdots \cup L_j \cup \cdots \cup L_m \\ &\qquad \equiv_{\mathrm{fbt}} L_1 \cup \cdots \cup L_{i-1} \cup L_{i+1} \cup \cdots \cup L_j \cup \cdots \cup L_m . \end{aligned}$$

COROLLARY 2. *Let $L_1, \ldots, L_i, \ldots, L_m$ be regularly separable, $L_1, \ldots, L_{i-1}, L_i', L_{i+1}, \ldots, L_m$ be regularly separable and $L_i \equiv_{\mathrm{fbt}} L_i'$. Then*

$$\begin{aligned} &L_1 \cup \cdots \cup L_i \cup \cdots \cup L_m \\ &\qquad \equiv_{\mathrm{fbt}} L_1 \cup \cdots \cup L_{i-1} \cup L_i' \cup L_{i+1} \cup \cdots \cup L_m . \end{aligned}$$

Let $\mathbf{N}$ denote the set of all nonnegative integers. Let $\alpha = (x_1, \ldots, x_n, y_0, y_1, \ldots, y_n)$ be a sequence of $2n + 1$ words $(n \geqslant 1)$. For each $h \geqslant 0$, let $\alpha^{(h)}$ denote $(x_1^h, \ldots, x_n^h, y_0, y_1, \ldots, y_n)$. (For a word u and $t > 0$, u^t denotes the word obtained by concatenating u t times. u^0 denotes ϵ.) For a language L, let L/α denote the set of n-tuples of nonnegative integers

$$\{(p_1, \ldots, p_n) \mid (p_1, \ldots, p_n) \text{ in } \mathbf{N}^n, y_0 x_1^{p_1} y_1 \cdots x_n^{p_n} y_n \text{ in } L\}.$$

THEOREM 3. *If $L_1 \leqslant_{\mathrm{fbt}} L_2$ and α is a sequence of $2n + 1$ words such that for each $h > 0$*

$$L_1/\alpha^{(h)} \cap (\mathbf{N} - \{0\})^n \neq \varnothing ,$$

then there exist $h_0 > 0$ *and a sequence* β *of* $2n + 1$ *words such that*

$$L_1/\alpha^{(h_0)} - (1, 1, \ldots, 1) = L_2/\beta,$$

where $L_1/\alpha^{(h_0)} - (1, 1, \ldots, 1)$ *denotes* $\{(p_1 - 1, p_2 - 1, \ldots, p_n - 1) \mid (p_1, p_2, \ldots, p_n)$ *in* $L_1/\alpha^{(h_0)} \cap (\mathbf{N} - \{0\})^n\}$.

Proof. Let $a_1, \ldots, a_n$ be n different letters, α_0 be the sequence of $2n + 1$ words $(a_1, \ldots, a_n, \epsilon, \epsilon, \ldots, \epsilon)$. Let L_0 be the language

$$\{a_1^{p_1} \cdots a_n^{p_n} \mid (p_1, \ldots, p_n) \text{ in } L_1/\alpha \cap (\mathbf{N} - \{0\})^n\}.$$

Then $L_0 \leqslant_{\text{fbt}} L_1 \leqslant_{\text{fbt}} L_2$ and for each $h > 0$,

$$L_0/\alpha_0^{(h)} = L_1/\alpha^{(h)} \cap (\mathbf{N} - \{0\})^n.$$

Let $A = (S, \Sigma_1, \Sigma_2, \delta, s_0, F)$ be a generalized 2NDFA such that $R(A)$ is a functional binary transduction and $L_0 \leqslant_{\text{fbt}} L_2$ (via $R(A)$). We may suppose that ϵ is not in domain $(R(A))$ because ϵ is not in L_0. Therefore, we may suppose that each element of δ is of the form $(s, \{a\}, N, s')$, where s, s' are in S, a is in Σ_1 and N is a nonempty regular subset of $\Sigma_2{}^*$.

For each s, s' in S and $a_i (1 \leqslant i \leqslant n)$, let $P(s, a_i, s')$ be the set of all paths from s to s' of which a word of the form $a_i{}^t (t \geqslant 1)$ is an input word. If we regard δ as an alphabet, the set $P(s, a_i, s')$ is a regular subset of δ^*. Let $D(s, a_i, s')$ be the set of lengths of paths in $P(s, a_i, s')$. $D(s, a_i, s')$ is an ultimately periodic set. When $D(s, a_i, s')$ is infinite, let $d(s, a_i, s')$, $e(s, a_i, s')$ be two arbitrary positive integers such that, for each $d \geqslant d(s, a_i, s')$, d is in $D(s, a_i, s')$ if and only if $d + e(s, a_i, s')$ is in $D(s, a_i, s')$.

Let g' be an arbitrary positive integer such that, if $D(s, a_i, s')$ is finite for each s, s' in S and $a_i (1 \leqslant i \leqslant n)$, then any length in $D(s, a_i, s')$ is less than g'. Let g be the positive integer $g' \prod d(s, a_i, s')\, e(s, a_i, s')$, where the multiplication $\prod$ ranges over all s, s' in S and $a_i (1 \leqslant i \leqslant n)$ such that $D(s, a_i, s')$ is infinite.

By the assumption $L_0/\alpha_0^{(g)} = L_1/\alpha^{(g)} \cap (\mathbf{N} - \{0\})^n \neq \varnothing$. Let $(r_1, \ldots, r_n)$ be an element of $L_0/\alpha_0^{(g)}$. $a_1^{r_1 g} \cdots a_n^{r_n g}$ is in L_0 and consequently in domain $(R(A))$. Hence there exist states $s_{j_0}, s_{j_1}, \ldots, s_{j_n}$ such that $s_{j_0} = s_0$, s_{j_n} is in F, and for each $i (1 \leqslant i \leqslant n)$ $r_i g$ is in $D(s_{j_{i-1}}, a_i, s_{j_i})$. From the definition of g' and the inequality $r_i g \geqslant g'$, $D(s_{j_{i-1}}, a_i, s_{j_i})$ must be infinite. Then $p r_i g$ must be in $D(s_{j_{i-1}}, a_i, s_{j_i})$ for each $p \geqslant 1$ because (1) $p r_i g = r_i g + (p - 1) r_i g$, (2) $r_i g \geqslant d(s_{j_{i-1}}, a_i, s_{j_i})$, (3) $r_i g$ is in $D(s_{j_{i-1}}, a_i, s_{j_i})$, (4) $(p - 1) r_i g$ is divisible by $e(s_{j_{i-1}}, a_i, s_{j_i})$. In other words, for each $p \geqslant 1$ the regular set $P(s_{j_{i-1}}, a_i, s_{j_i})$ contains a path whose length is a positive multiple of p.

Hence, by Lemma 1 in Kobayashi (1969), there is a path $w_{i1}w_{i2}w_{i3}$ in $P(s_{j_{i-1}}, a_i, s_{j_i})$ such that $|w_{i1}| + |w_{i3}| = |w_{i2}| > 0$ and $w_{i1}w_{i2}^p w_{i3}$ is a path in $P(s_{j_{i-1}}, a_i, s_{j_i})$ for each $p \geqslant 0$. Let $v'_{i1}, v'_{i2}, v'_{i3}$ be output words of the paths w_{i1}, w_{i2}, w_{i3}, respectively, and $h_0 = |w_{12}| \cdot |w_{22}| \cdots |w_{n2}|$ (>0), $v_{i1} = v'_{i1}$, $v_{i2} = v'^{h_0/|w_{i2}|}_{i2}$, $v_{i3} = v'^{h_0/|w_{i2}|-1}_{i2} v'_{i3}$. For each $p_i > 0$, $w_{i1}w_{i2}^{p_i h_0/|w_{i2}|-1} w_{i3}$ is a path in $P(s_{j_{i-1}}, a_i, s_{j_i})$. a_i^t is an input word of this path, where

$$t = |w_{i1}| + (p_i h_0/|w_{i2}| - 1) \cdot |w_{i2}| + |w_{i3}| = p_i h_0,$$

and

$$v'_{i1} v'^{p_i h_0/|w_{i2}|-1}_{i2} v'_{i3} = v_{i1} v_{i2}^{p_i - 1} v_{i3}$$

is an output word of the path. Hence, for each $(p_1, \ldots, p_n)$ in $(\mathbf{N} - \{0\})^n$, there is a path of A from the initial state $s_0 = s_{j_0}$ to a state s_{j_n} in F of which $a_1^{p_1 h_0} \cdots a_n^{p_n h_0}$ is an input word and $v_{11}v_{12}^{p_1-1}v_{13} \cdots v_{n1}v_{n2}^{p_n-1}v_{n3}$ is an output word. In other words, for each $(p_1, \ldots, p_n)$ in $(\mathbf{N} - \{0\})^n$, $(a_1^{p_1 h_0} \cdots a_n^{p_n h_0}, v_{11}v_{12}^{p_1-1}v_{13} \cdots v_{n1}v_{n2}^{p_n-1}v_{n3})$ is in $R(A)$. Let $\beta = (v_{12}, \ldots, v_{n2}, v_{11}, v_{13}v_{21}, \ldots, v_{n-1,3}v_{n1}, v_{n3})$. Then

$$\begin{aligned}
L_1/\alpha^{(h_0)} &- (1, 1, \ldots, 1) \\
&= L_0/\alpha_0^{(h_0)} - (1, 1, \ldots, 1) \\
&= \{(p_1 - 1, \ldots, p_n - 1) \mid (p_1, \ldots, p_n) \text{ in } (\mathbf{N} - \{0\})^n, \\
&\qquad a_1^{p_1 h_0} \cdots a_n^{p_n h_0} \text{ in } L_0\} \\
&= \{(p_1 - 1, \ldots, p_n - 1) \mid (p_1, \ldots, p_n) \text{ in } (\mathbf{N} - \{0\})^n, \\
&\qquad v_{11}v_{12}^{p_1-1}v_{13} \cdots v_{n1}v_{n2}^{p_n-1}v_{n3} \text{ in } L_2\} \\
&= \{(q_1, \ldots, q_n) \mid (q_1, \ldots, q_n) \text{ in } \mathbf{N}^n, \\
&\qquad v_{11}v_{12}^{q_1}v_{13} \cdots v_{n1}v_{n2}^{q_n}v_{n3} \text{ in } L_2\} \\
&= L_2/\beta.
\end{aligned}$$

This completes the proof.

2. Minimal fbt-Degrees in the fbt-Degrees of Nonregular CFL's

In Kobayashi (1969), we have shown that the fbt-degree $\gamma_{\varnothing}$ consisting of the empty set only is the least fbt-degree in all fbt-degrees, and that the fbt-degree γ_{reg} consisting of all nonempty regular sets is the least fbt-degree in the fbt-degrees of nonempty languages. These results naturally give rise to the following problem: Is there a least fbt-degree in the set of all fbt-degrees of nonregular context-free languages? In other words, is there a nonregular context-free language L_0 that is structurally so simple that L_0 is fbt-translatable to any nonregular context-free language? In the following theorem we solve this problem negatively by showing that there are denumerably infinite minimal fbt-degrees in the fbt-degrees of nonregular context-free languages.

THEOREM 4. *There are denumerably infinite* fbt-*degrees* $\gamma_1, \gamma_2, \ldots$ *of nonregular context-free languages such that, for each* γ *and* $1 \leqslant i$, *if* $\gamma < \gamma_1$, *then either* $\gamma = \gamma_\varnothing$ *or* $\gamma = \gamma_{\text{reg}}$, *where* $\gamma_\varnothing$ *is the* fbt-*degree consisting of the empty set only and* γ_{reg} *is the* fbt-*degree consisting of all nonempty regular sets.*

Proof. For each integer $k \geqslant 2$ let $L_k = \{a^p b^q \mid 0 \leqslant p \leqslant q \leqslant kp\}$, where a, b are two different letters, and let γ_{k-1} be the fbt-degree of L_k. Evidently, γ_{k-1} is an fbt-degree of nonregular context-free languages.

First we show that if $k \neq k'$ then neither $L_k \leqslant_{\text{fbt}} L_{k'}$ nor $L_{k'} \leqslant_{\text{fbt}} L_k$. Suppose that $k \neq k'$ and $L_k \leqslant_{\text{fbt}} L_{k'}$. Let $\alpha = (a, b, \epsilon, \epsilon, \epsilon)$. For any $h > 0$, $L_k/\alpha^{(h)} = \{(p,\ q) \mid 0 \leqslant hp \leqslant hq \leqslant khp\} = \{(p,\ q) \mid 0 \leqslant p \leqslant q \leqslant kp\}$ and $L_k/\alpha^{(h)} \cap (\mathrm{N} - \{0\})^2 \neq \varnothing$. By Theorem 3, there exist $h_0 > 0$ and $\beta = (x_1, x_2, y_0, y_1, y_2)$ such that $L_k/\alpha^{(h_0)} - (1, 1) = L_{k'}/\beta$. But

$$\begin{aligned} L_k/\alpha^{(h_0)} - (1, 1) &= \{(p-1, q-1) \mid 0 < p, 0 < q, 0 \leqslant p \leqslant q \leqslant kp\} \\ &= \{(p, q) \mid 0 \leqslant p \leqslant q \leqslant kp + (k-1)\}. \end{aligned}$$

Hence we have the following equation:

$$\begin{aligned} &\{(p, q) \mid 0 \leqslant p \leqslant q \leqslant kp + (k-1)\} \\ &\quad = \{(p, q) \mid 0 \leqslant p, 0 \leqslant q, y_0 x_1{}^p y_1 x_2{}^q y_2 \text{ in } L_{k'}\}. \end{aligned}$$

A minute observation shows that this is possible if and only if (1) $k = k'$, (2) there exist $d > 0$, $e_1 \geqslant 0$, $e_2 \geqslant 0$ such that $-1 < (e_1 - e_2)/d \leqslant 0$, $k - 1 \leqslant (e_1 k - e_2)/d < k$ and $y_0 x_1{}^p y_1 x_2{}^q y_2 = a^{dp+e_1} b^{dq+e_2}$ for each $p \geqslant 0$, $q \geqslant 0$. This contradicts the assumption $k \neq k'$. Hence $L_k \leqslant_{\text{fbt}} L_{k'}$ is impossible. Similarly, $L_{k'} \leqslant_{\text{fbt}} L_k$ is impossible. This implies that $\gamma_1, \gamma_2, \ldots$ are in fact different fbt-degrees.

Next we show that, for each $k \geqslant 2$, if M is a regular set then $M \cap L_k$ is the union of some regularly separable sets $W_1, W_2, \ldots, W_m (m \geqslant 1)$ such that for each $i (1 \leqslant i \leqslant m)$ either W_i is regular or $L_k \leqslant_{\text{fbt}} W_i$. Let M be an arbitrary regular set and let $A = (S, \Sigma_1, s_0, \delta, F)$ be a finite automation accepting M, where S is the set of states, Σ_1 is the alphabet of inputs, s_0 is the initial state, δ is the transition function from $S \times \Sigma_1$ into S and F ($\subseteq S$) is the set of accepting states. δ is naturally extended to a function from $S \times \Sigma_1{}^*$ to S by $\delta(s, \epsilon) = s$, $\delta(s, ua_1) = \delta(\delta(s, u), a_1)$ for each s in S, u in $\Sigma_1{}^*$ and a_1 in Σ_1. M is, by definition, the set $\{u \mid \delta(s_0, u) \text{ is in } F\}$. Let s be the i-th state in S and let $M(s) = \{a^p \mid p \geqslant 0,\ \delta(s_0, a^p) = s\}$, $M'(s) = \{b^q \mid q \geqslant 0,\ \delta(s, b^q) \text{ in } F\}$, $W_i = M(s)\, M'(s) \cap L_k$ ($1 \leqslant i \leqslant m$, where m is the number of states in S). Then $M \cap L_k$ is the union of these regularly separable sets W_i. If either $M(s)$ or $M'(s)$ is finite, then $W_i = M(s)\, M'(s) \cap L_k$ is finite and, consequently, regular. Suppose that both $M(s)$ and $M'(s)$ are infinite. Then there exist d_1, e_1, d_2, e_2 such that $d_1 > 0$, $e_1 \geqslant 0$, $d_2 > 0$, $e_2 \geqslant 0$, $\{a^{d_1p+e_1}b^{d_2q+e_2} \mid 0 \leqslant p,\ 0 \leqslant q\} \subseteq M(s)\, M'(s)$. By the assumption, $k \geqslant 2$. Hence we can select integers $p_0 \geqslant 0$, $q_0 \geqslant 0$, $t > 0$ such that $0 \leqslant d_1p_0 + e_1 \leqslant d_2q_0 + e_2 \leqslant k(d_1p_0 + e_1)$, $d_1p_0 + td_1d_2 + e_1 > d_2q_0 + e_2$, $d_2q_0 + td_1d_2 + e_2 > k(d_1p_0 + e_1)$. Let f be the functional binary transduction $\{(a^pb^q,\ a^{d_1p_0+ptd_1d_2+e_1}b^{d_2q_0+qtd_1d_2+e_2}) \mid p \geqslant 0,\ q \geqslant 0\}$. It is easily shown that $L_k \leqslant_{\text{fbt}} M(s)\, M'(s) \cap L_k = W_i$ (via f).

Let γ be an fbt-degree such that $\gamma <_{\text{fbt}} \gamma_k$. By Theorem 1, there exists a regular set M such that $M \cap L_{k+1}$ is in γ. Let $M \cap L_{k+1}$ be decomposed to regularly separable sets W_1, W_2,..., W_m ($m \geqslant 1$) by the above described method. If there exists W_i such that $L_{k+1} \leqslant_{\text{fbt}} W_i$, then $L_{k+1} \leqslant_{\text{fbt}} W_i \leqslant_{\text{fbt}} M \cap L_{k+1}$ by Theorem 2. This contradicts $\gamma < \gamma_k$. Hence all W_1, W_2,..., W_m are regular. If all of them are empty, $\gamma = \gamma_\varnothing$. If at least one of them is nonempty, $\gamma = \gamma_{\text{reg}}$. This completes the proof.

Corollary 3. *There is no least* fbt-*degree in the set of all* fbt-*degrees of nonregular context-free languages.*

Corresponding to Corollary 3, the following problem arises naturally: Is there a greatest fbt-degree in the set of all fbt-degrees of context-free languages? This problem is still open although the negative solution seems to be plausible.

3. fbt-Degrees Having No Greatest Lower Bound

In the partially ordered set Γ of all fbt-degrees having the fbt-translatability as the partial ordering, any two fbt-degrees γ_1, γ_2 have a least upper bound. In fact, if L_1, L_2 are languages in γ_1, γ_2, respectively, and L_1, L_2 are regularly separable (such languages L_1, L_2 always exist), then the fbt-degree of $L_1 \cup L_2$ is the least upper bound of γ_1, γ_2. Hence the set Γ is an upper semilattice.

The purpose of this section is to prove that the set Γ is not a lattice by showing two fbt-degrees of context-free languages that have no greatest lower bound.

Let a, a', b, b', c, c' be different letters and let Z_1, Z_2, X, Y be the context-free languages defined as follows:

$$Z_1 = \{a^p a'^p \mid p \geqslant 1\},$$

$$Z_2 = \{b^p c^q c'^q b'^p \mid p, q \geqslant 1\},$$

$$X = \bigcup_{1 \leqslant m \leqslant n} Z_1{}^m Z_2 Z_1{}^n,$$

$$Y = \bigcup_{m \geqslant n \geqslant 1} Z_1{}^m Z_2 Z_1{}^n.$$

LEMMA 1. $\{a^p a'^q \mid 1 \leqslant p \leqslant q\} \not\leqslant_{\mathrm{fbt}} Y$.

Proof. Let $L_0 = \{a^p a'^q \mid 1 \leqslant p \leqslant q\}$ and suppose that $L_0 \leqslant_{\mathrm{fbt}} Y$. Let $\alpha = (a, a', \epsilon, \epsilon, \epsilon)$. Then for each $h > 0$ $L_0/\alpha^{(h)} = \{(p, q) \mid 1 \leqslant p \leqslant q\}$ and $L_0/\alpha^{(h)} \cap (\mathbf{N} - \{0\})^2 \neq \varnothing$. Hence by Theorem 3 there exist $h_0 > 0$ and $\beta = (x_1, x_2, y_0, y_1, y_2)$ such that

$$\begin{aligned}
&\{(p, q) \mid 0 \leqslant p \leqslant q\} \\
&\quad = L_0/\alpha^{(h_0)} - (1, 1) \\
&\quad = Y/\beta \\
&\quad = \{(p, q) \mid 0 \leqslant p, 0 \leqslant q, y_0 x_1{}^p y_1 x_2{}^q y_2 \text{ in } Y\}.
\end{aligned}$$

From (i) $y_0 y_1 y_2$ in Y, (ii) $y_0 x_1 y_1 y_2$ not in Y, (iii) $y_0 x_1 y_1 x_2 y_2$ in Y, x_1 and x_2 cannot be ϵ. From (iv) $y_0 x_1{}^p y_1 x_2{}^p y_2$ in Y for each $p \geqslant 0$, each of x_1 and x_2 must be a word of one of the following forms: (1) a^t, (2) a'^t, (3) b^t, (4) c^t, (5) c'^t, (6) b'^t ($t \geqslant 1$ for (1)–(6)), (7) $u_1 u_2 u_3$, where u_2 is in $Z_1{}^*$, $u_3 u_1$ is in Z_1, $u_1 \neq \epsilon$. But a minute observation shows that for each x_1, x_2 of the forms (1)–(7) the equality $\{(p, q) \mid 0 \leqslant p \leqslant q\} = \{(p, q) \mid 0 \leqslant p, 0 \leqslant q, y_0 x_1{}^p y_1 x_2{}^q y_2$ in $Y\}$ cannot hold true. This completes the proof.

LEMMA 2.[1] $Z_1^{m+1} Z_2 \not\leqslant_{\mathrm{fbt}} Z_1{}^m Z_2 Z_1{}^+ (m \geqslant 0)$.

Proof. To simplify the description, we prove for the case $m = 1$ only. Let $\alpha = (a, a', a, a', b, c, c', b', \epsilon, \epsilon, \ldots, \epsilon)$ (nine ϵ's). Then, for each $h > 0$, $Z_1{}^2 Z_2/\alpha^{(h)} = \{(p_1, p_1, p_2, p_2, p_3, p_4, p_4, p_3) \mid p_1, p_2, p_3, p_4 \geqslant 1\}$

[1] For each language L, let L^+ denote the language $L \cup L^2 \cup L^3 \cup \cdots$. For each letter a_1, let $a_1{}^+$ denote $\{a_1\}^+$.

and $Z_1{}^2Z_2/\alpha^{(h)} \cap (\mathbf{N} - \{0\})^8 \neq \varnothing$. Hence by Theorem 3 there exist $h_0 > 0$ and $\beta = (x_1, x_2, \ldots, x_8, y_0, y_1, \ldots, y_8)$ such that

$$\begin{aligned}\{(p_1, p_1, p_2, p_2, p_3, p_4, p_4, p_3) \mid p_1, p_2, p_3, p_4 \geqslant 0\} \\ = Z_1{}^2Z_2/\alpha^{(h_0)} - (1, 1, \ldots, 1) \\ = Z_1Z_2Z_1{}^+/\beta.\end{aligned}$$

By arguments similar to those in the proof of Lemma 1, each $x_i(1 \leqslant i \leqslant 8)$ must be a word of the forms (1)–(7) mentioned there. But a minute observation shows that for each $x_i(1 \leqslant i \leqslant 8)$ of the forms (1)–(7), the equality $\{(p_1, p_1, p_2, p_2, p_3, p_4, p_4, p_3) \mid p_1, p_2, p_3, p_4 \geqslant 0\} = Z_1Z_2Z_1{}^+/\beta$ cannot hold true. This completes the proof.

THEOREM 5. *There exist two* fbt-*degrees of context-free languages that have no greatest lower bound with respect to the* fbt-*translatability relation.*

Proof. Let γ_1, γ_2 be the fbt-degrees of X, Y respectively. They are fbt-degrees of context-free languages. We show that these two fbt-degrees have no greatest lower bound. Suppose that γ is a greatest lower bound of γ_1, γ_2. By $\gamma \leqslant_{\text{fbt}} \gamma_1$ and Theorem 1, there exists a regular set M such that $M \cap X$ is in γ. Let $A = (S, \Sigma_1, s_0, \delta, F)$ be a finite automaton accepting M. For each s, s' in S, let

$$\begin{aligned} M_1(s) &= \{u \mid \delta(s_0, u) = s,\ u \text{ in } (a^+a'^+)^+\}, \\ M_2(s, s') &= \{u \mid \delta(s, u) = s',\ u \text{ in } b^+c^+c'^+b'^+\}, \\ M_3(s') &= \{u \mid \delta(s', u) \text{ in } F,\ u \text{ in } (a^+a'^+)^+\}. \end{aligned}$$

Then $M \cap X$ is the union of regularly separable sets $M_1(s)\, M_2(s, s')\, M_3(s') \cap X(s, s' \text{ in } S)$. We now show that for each s, s' in S there exists an integer $m(s, s') \geqslant 0$ such that $M_1(s)\, M_2(s, s')\, M_3(s') \cap X \leqslant_{\text{fbt}} Z_1^{m(s,s')}Z_2Z_1{}^+$. If $M_1(s)\, M_2(s, s')\, M_3(s') \cap X = \varnothing$ then this is evident. Suppose that $M_1(s)\, M_2(s, s')\, M_3(s') \cap X \neq \varnothing$. We consider three cases.

Case 1. There exists m_0 such that for all $m > m_0$, $M_1(s) \cap Z_1{}^m = \varnothing$. In this case, let $m(s, s') = m_0$. Then

$$\begin{aligned} & M_1(s)\, M_2(s, s')\, M_3(s') \cap X \\ &= M_1(s)\, M_2(s, s')\, M_3(s') \cap \Bigg(\bigcup_{\substack{1 \leqslant m \leqslant n \\ m \leqslant m_0}} Z_1{}^mZ_2Z_1{}^n \Bigg) \\ &\leqslant_{\text{fbt}} \bigcup_{\substack{1 \leqslant m \leqslant n \\ m \leqslant m_0}} Z_1{}^mZ_2Z_1{}^n \qquad \text{(by Theorem 1)} \\ &\leqslant_{\text{fbt}} Z_1^{m_0}Z_2Z_1{}^+ = Z_1^{m(s,s')}Z_2Z_1{}^+. \end{aligned}$$

Case 2. There exists n_0 such that for all $n > n_0$, $M_3(s') \cap Z_1{}^n = \varnothing$. In this case, let $m(s, s') = n_0$. Then

$$\begin{aligned}
&M_1(s)\, M_2(s, s')\, M_3(s') \cap X \\
&\quad = M_1(s)\, M_2(s, s')\, M_3(s') \cap \Bigg(\bigcup_{\substack{1 \leqslant m \leqslant n \\ n \leqslant n_0}} Z_1{}^m Z_2 Z_1{}^n \Bigg) \\
&\quad \leqslant_{\text{fbt}} \bigcup_{\substack{1 \leqslant m \leqslant n \\ n \leqslant n_0}} Z_1{}^m Z_2 Z_1{}^n \qquad \text{(by Theorem 1)} \\
&\quad \leqslant_{\text{fbt}} Z_1^{n_0} Z_2 Z_1{}^+ = Z_1^{m(s,s')} Z_2 Z_1{}^+.
\end{aligned}$$

Case 3. Neither Case 1 nor Case 2 holds true. In this case there exist words $u_1, u_2, \ldots, u_m$ such that (1) each u_i is in Z_1, (2) $u_1 \cdots u_m$ is in $M_1(s)$, (3) $m \geqslant |S|$, where $|S|$ is the number of states in S. Then there exist $g \geqslant 0$, $d_2 > 0$ such that $g + d_2 \leqslant m$, $\delta(s_0, u_1 \cdots u_g) = \delta(s_0, u_1 \cdots u_g u_{g+1} \cdots u_{g+d_2})$. Let $d_1 = m - d_2 (\geqslant 0)$, $x_1 = u_1 \cdots u_g$, $x_2 = u_{g+1} \cdots u_{g+d_2}$, $x_3 = u_{g+d_2+1} \cdots u_m$. Then, for each $p \geqslant 0$, $x_1 x_2{}^p x_3$ is in $M_1(s) \cap Z_1^{d_1+d_2 p}$. Similarly, there exist $e_1 \geqslant 0$, $e_2 > 0$, z_1, z_2, z_3 such that, for each $q \geqslant 0$, $z_1 z_2{}^q z_3$ is in $M_3(s') \cap Z_1^{e_1+e_2 q}$. By the assumption $M_1(s)\, M_2(s, s')\, M_3(s') \cap X \neq \varnothing$, $M_2(s, s') \cap Z_2$ contains a word y. Let p_0, q_0, t be integers such that $d_1 + d_2 p_0 \leqslant e_1 + e_2 q_0$, $d_1 + d_2 p_0 + d_2 e_2 t > e_1 + e_2 q_0$. Let f be the functional binary transduction $\{(a^p a'^q, x_1 x_2^{p_0+e_2 t p} x_3 y z_1 z_2^{q_0+d_2 t q} z_3) \mid 1 \leqslant p, q\}$. It is easily shown that $\{a^p a'^q \mid 1 \leqslant p \leqslant q\} \leqslant_{\text{fbt}} M_1(s)\, M_2(s, s')\, M_3(s') \cap X$ (via f). Then,

$$\begin{aligned}
\{a^p a'^q \mid 1 \leqslant p \leqslant q\} &\leqslant_{\text{fbt}} M_1(s)\, M_2(s, s')\, M_3(s') \cap X \\
&\leqslant_{\text{fbt}} M \cap X \qquad \text{(by Theorem 2)} \\
&\leqslant_{\text{fbt}} Y \qquad\quad\;\; \text{(by } \gamma \leqslant_{\text{fbt}} \gamma_2\text{)}.
\end{aligned}$$

This contradicts Lemma 1. Hence Case 3 is impossible.

Thus we have shown that for each s, s' in S there exists an integer $m(s, s') \geqslant 0$ such that $M_1(s)\, M_2(s, s')\, M_3(s') \cap X \leqslant_{\text{fbt}} Z_1^{m(s,s')} Z_2 Z_1{}^+$. Let $t = \max\{m(s, s') \mid s, s' \text{ in } S\}$. Then for each s, s' in S, $Z_1^{m(s,s')} Z_2 Z_1{}^+ \leqslant_{\text{fbt}} Z_1{}^t Z_2 Z_1{}^+$. Therefore, by Theorem 2, $M \cap X \leqslant_{\text{fbt}} Z_1{}^t Z_2 Z_1{}^+$. By directly constructing functional binary transductions we can show $Z_1^{t+1} Z_2 \leqslant_{\text{fbt}} X$ and $Z_1^{t+1} Z_2 \leqslant_{\text{fbt}} Y$. Hence the fbt-degree of $Z_1^{t+1} Z_2$ is a lower bound of γ_1, γ_2. Since the fbt-degree of $M \cap X$ is the greatest lower bound, $Z_1^{t+1} Z_2 \leqslant_{\text{fbt}} M \cap X \leqslant_{\text{fbt}} Z_1{}^t Z_2 Z_1{}^+$. This contradicts Lemma 2. Hence γ_1, γ_2 have no greatest lower bound. This completes the proof.

COROLLARY 4. *The* fbt-*translatability relation in the set of all* fbt-*degrees of context-free languages is not a lattice.*

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